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THE INSTABILITY OF
LINEAR HETEROGENEOUS LANCHESTER EQUATIONS

HERBERT E. COHEN

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U. S. ARMY MATERIEL SYSTEMS ANALYSIS ACTIVITY
ABERDEEN PROVING GROUND, MARYLAND

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It is shown that the structure of the linear Lanchester attrition matrix leads to invariant properties associated with the sum of the eigenvalues and to the conclusion the linear Lanchester equations are inherently unstable.

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THE INSTABILITY OF LINEAR HETEROGENEOUS LANCHESTER EQUATIONS

Invariant properties of differential equations provide useful information on the nature of the solutions. The linear Lanchester equations have interesting invariant properties which this writer believes are not readily known within the community.

An n-dimensional linear time varying Lanchester equation [1] can be described by

$$\dot{x} = F(t)\ x(t)\ ,\ x(t_o) \qquad (1)$$

where F(t) is a nxn matrix in R and can be represented by a partitioned matrix of the form

$$F = \begin{pmatrix} \underline{0} & -A \\ -B & \underline{0} \end{pmatrix}$$

where the elements of F, $[f_{ij}]$, are all less than 1 and x is a n-dimensional "force strength" vector in R^n and t_o is the initial time. Introducing the fundamental matrix $\Phi(t,t_o)$ [2-8] with the property

$$x(t) = \phi\ (t, t_o)\ x\ (t_o)$$

it can be shown that the

$$\det\ \phi\ (t,\ t_o) = e^{\int_{to}^{t} tr\ F(t)\, dt} \qquad (2)$$

Since the trace of F, tr (F(t)), is zero for all t, it follows that

$$\det\ \phi(t,\ t_o) = 1 \qquad (3)$$

for all n-dimension linear Lanchester equations.

Since in many cases in practice, $F(t)$, is taken as constant or piecewise constant over a sufficiently small intervals $t_i \leq t \leq t_{i+1}$, it follows from systems theory that

$$\Phi(t, t_o) = T^{-1} e^{Dt} T \tag{4}$$

where T is a similarity transformation and D is a diagonal matrix of eigenvalues for $F(t)$ in $t_i \leq t \leq t_{i+1}$, assuming distinct eigenvalues. Thus, it follows that the

$$\det \Phi(t, t_o) = \det(e^{Dt})$$
$$= e^{\left(\sum \lambda_i\right)t}$$

Since $\det \Phi(t, t_o) = 1$ we can conclude that

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{5}$$

for all linear Lanchester equations. It is this property on the sum of the eigenvalues which is invariant for linear Lanchester equations. It should be noted that Eq (5) is also true when the eigenvalues are <u>not</u> distinct.

The analysis of Equation (5) leads one to conclude that either all eigenvalues reside on the imaginary axis resulting in harmonic oscillation or there exists at least one eigenvalue residing in the right half plane so that our system is inherently <u>unstable</u>.

When one applies a finite difference approximation to the derivative in Eq (1) we obtain

$$x(t_i + 1) = (I + F(t_i))\, x(t_i) \tag{6}$$

It then follows that the sum of the eigenvalues (ζ_i), associated with the difference equation, Eq (6), is

$$\sum_{i=1}^{n} \zeta_i = n \qquad (7)$$

where n is the dimension of x.

For Eq (6) to be stable all eigenvalues must lie in the unit circle so that $|\zeta_i| < 1$, otherwise it is unstable. The eigenvalues ζ_i associated with Eq (6) are related to the eigenvalues λ_i of the continuous case by

$$\zeta_i = 1 + \lambda_i$$

so that if the continuous problem is nonstable or have all of its eigenvalues on the imaginary axis, then the difference equation is inherently <u>unstable</u>.

It is appropriate at this point to present two examples which demonstrate this invariant property:

Case 1: Linear Lanchester equation of order two. This is given by the simple equation

$$\dot{x}_1 = -\alpha x_2$$

$$\dot{x}_2 = -\beta x_1$$

$$or \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which for $x = (x_1, x_2)^T$ *we have*

$$\dot{x} = Fx$$

$$with \quad F = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}$$

and α, β real and less than one.

The matrix F has eigenvalues

$$\lambda_+ = +\sqrt{\alpha\ \beta}$$

and

$$\lambda_- = -\sqrt{\alpha\ \beta}$$

so that the 2nd order system in inherently <u>unstable</u>. Similarly the corresponding difference equation will also be unstable since it has an eigenvalue of value

$1 + \sqrt{\alpha\ \beta}$ *which is outside the unit circle.*

Case 2: Linear Lanchester equation of nth order (n=50). This equation is of the form

$$\dot{x} = Fx$$

where F is 50 x 50 and represented

$$\text{by } F = \begin{pmatrix} \underline{0} & -A \\ -B & \underline{0} \end{pmatrix}$$

with A and B each being a 25 x 25 matrix whose elements are randomly generated by Matlab and less than one. This means that x is a vector of dimension 50. Using Matlab for the continuous case, it is found that

$$\sum_{i=1}^{50} \lambda_i = 1.3 \times 10^{-14}$$

with eigenvalues in the complex plane as shown in figure (1). It is immediately seen that λ_i have several eigenvalues in the right half plane so that this system is inherently <u>unstable</u>.

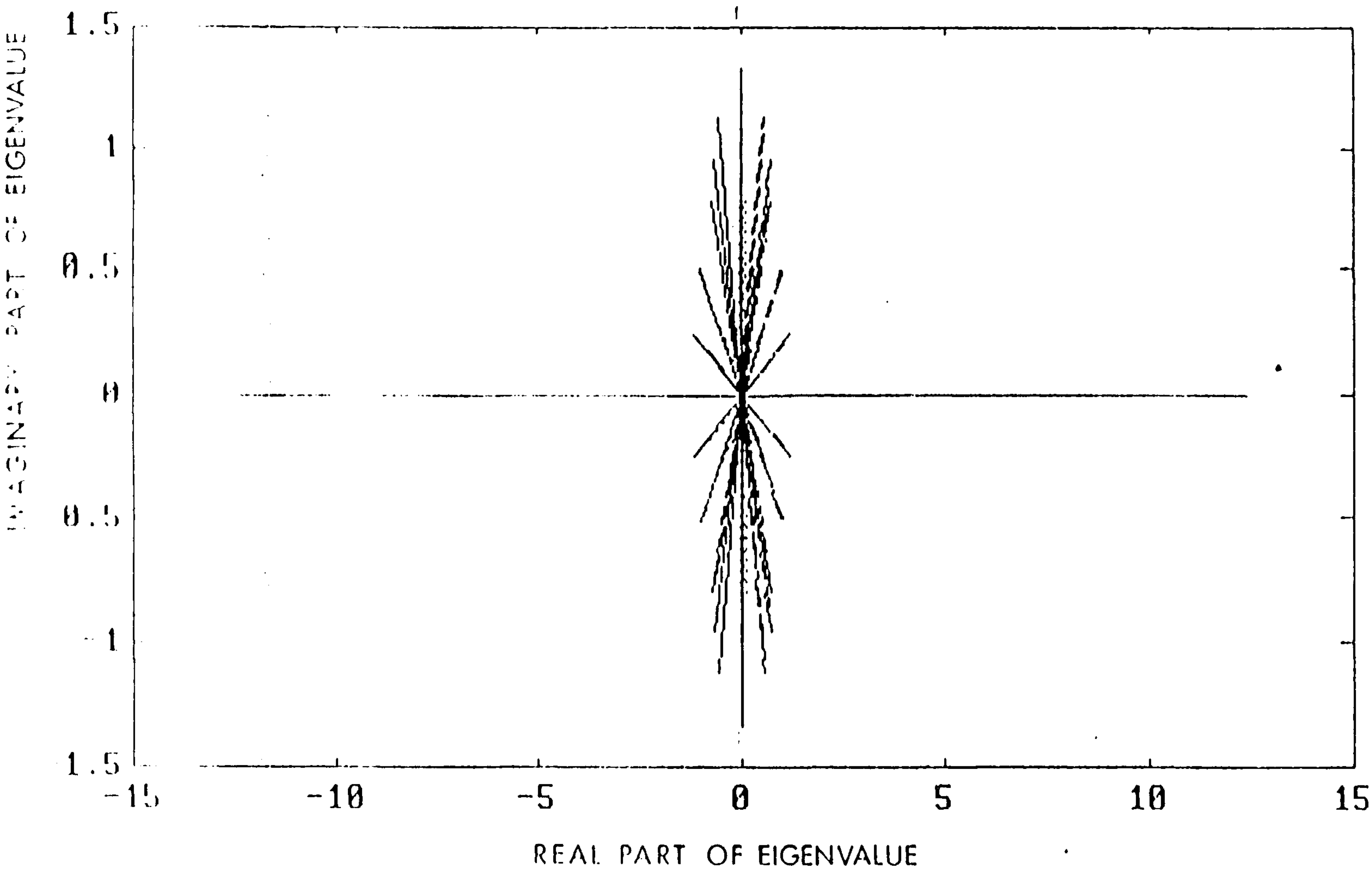
The finite difference equation for this problem yielded

$$\sum_{i=1}^{50} \zeta_i = 50$$

consistent with our analysis and has its eigenvalues in the complex plane as shown in figure (2). Please note the scales in both figures are different. However, there are several eigenvalues lying outside the unit circle so that this difference equation is inherently <u>unstable</u>.

In conclusion, we have seen that the structure of the linear heterogeneous Lanchester equations leads to an invariant condition associated with the sum of the eigenvalues with solutions which are inherently unstable or oscillate sinusoidally.

Figure 1. Eigenvalues in Complex Plane.
(Continuous Case)



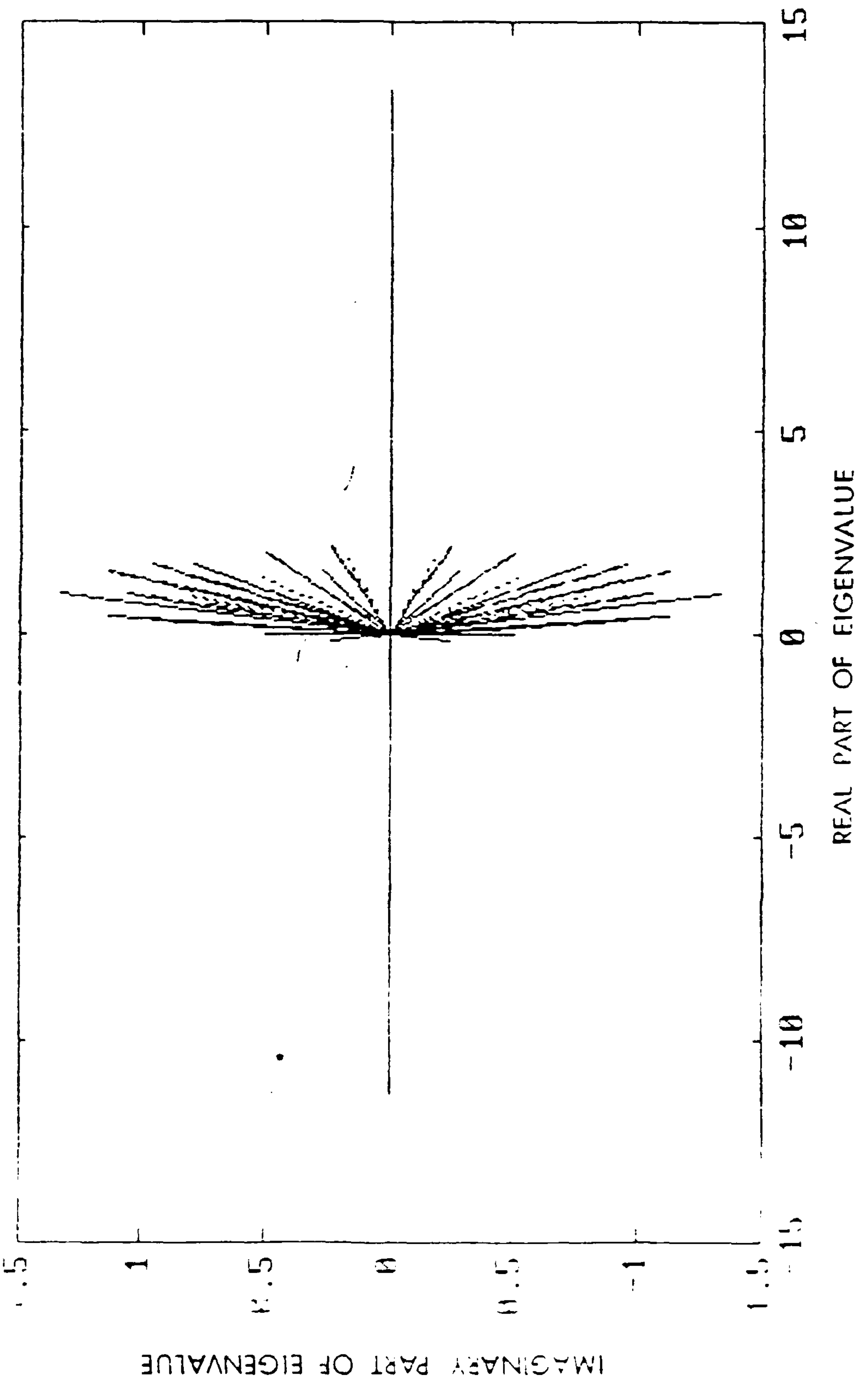


Figure 2. Eigenvalues in Complex Plane.
(Discrete Case)

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MAIN OBJECTIVES OF THE EFFORT: Investigate invariant properties of linear heterogeneous Lanchester equations of combat and determine the potential impact on combat models.

SCOPE OF THE EFFORT: Apply systems theory to linear heterogeneous Lanchester equations in state variables and introduce properties of the state transition matrix to develoop invariant properties of these equations.

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